29.5.2008

1

תרגיל: האם $\int_0^\infty \frac{dx}{1+(x\sin x)^2}$ מתכנס?

פתרון: $x \in [2\pi n, 2\pi(n+1)]$

$$I = \int_0^\infty \underbrace{\frac{dx}{1+(x\sin x)^2}}_{\text{חיובי}} = \sum_{n=0}^{\infty} \underbrace{\int_{2\pi n}^{2\pi(n+1)} \frac{dx}{1+(x\sin x)^2}}_{\geq \frac{1}{n}}$$

$$\int_{2\pi n}^{2\pi(n+1)} \frac{dx}{1+(x\sin x)^2} = \left\{ \begin{array}{l} t = x - 2\pi n \\ dt = dx \end{array} \right. = \int_0^{2\pi} \frac{dt}{1+(t+2\pi n)^2 \sin^2 t} \geq \quad (\sin t < t)$$

$$\geq \int_0^{2\pi} \frac{dt}{1+[2\pi(n+1)]^2 t^2} = \left\{ \begin{array}{l} S = 2\pi(n+1)t \\ dS = 2\pi(n+1)\,dt \end{array} \right. = \int_0^{(2\pi)^2(n+1)} \frac{1}{2\pi(n+1)} \cdot \frac{ds}{1+s^2} =$$

$$= \frac{1}{2\pi(n+1)} \underbrace{\arctan\Big|_0^{(2\pi)^2(n+1)}}_{\geq} \geq \overset{\text{קבוע}}{C} \cdot \frac{1}{n}$$

$$\arctan 1 - \arctan 0 = \arctan 1 \neq 0$$

$$\Rightarrow \quad I \geq C \cdot \sum \frac{1}{n} \to \infty$$

ולכן $\boxed{I \text{ מתבדר}}$

תרגיל: חשב את $I = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$

פתרון:

$$I = \int_0^{\frac{\pi}{4}} \log(\sin 2t)\, d(2t) = \int_0^{\frac{\pi}{4}} 2\log(\underset{2\sin t\cos t}{\sin 2t})\,dt = 2\int_0^{\frac{\pi}{4}} \Big(\log(\sin t) + \underbrace{\log(\cos t)}_{s=\frac{\pi}{2}-t} + \log 2\Big)\,dt =$$

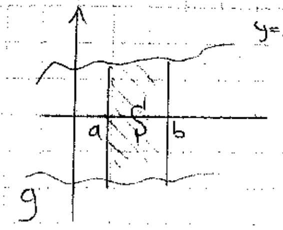
$$= \frac{\pi}{2}\log 2 - 2\int_0^{\frac{\pi}{4}} \log(\sin t)\,dt + 2\underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log\left(\cos\left(\tfrac{\pi}{2}-s\right)\right)\, d\left(\tfrac{\pi}{2}-s\right)}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log(\sin S)\,dS \qquad = \frac{\pi}{2}\log + 2\underbrace{\int_0^{\frac{\pi}{2}} \log(\sin t)\,dt}_{I}$$

$$I = \frac{\pi}{2}\log 2 + 2I$$

$$\Rightarrow \quad I = -\frac{\pi}{2}\log 2$$

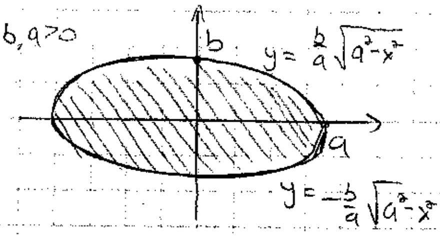
חשוב של שטח החסום ע"י העקומה



$$S = \int_a^b \left(f(x) - g(x)\right) dx$$

תרגיל: חשבו את שטח האליפסה, כלומר שטח החסום ע"י העקומה:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad b,a>0$$



$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$$|x| \le a$$

$$-a \le x \le a$$

והתחום חסום ע"י

$$f(x) = \frac{b}{a}\sqrt{a^2-x^2}$$

$$g(x) = -\frac{b}{a}\sqrt{a^2-x^2}$$

$$f(x) \ge g(x)$$

לכן:

$$S = \int_{-a}^{a} 2\frac{b}{a}\sqrt{a^2-x^2}\, dx = \begin{cases} x = at \\ dx = a\,dt \end{cases} = \frac{2b}{a}\int_{-1}^{1}\sqrt{a^2-a^2t^2}\; a\,dt =$$

$$= 2ab\int_{-1}^{1}\sqrt{1-t^2}\, dt = \begin{cases} t = \sin s \\ dt = \cos s\, ds \end{cases} = 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(s)\,ds =$$

$$= ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2s)\,ds = \pi ab$$

הערה: אם $a=b$ מקבלים מעגל והשטח שלו $\underline{\pi a^2}$